Dynamic Network Flows with Adaptive Route Choice based on Current Information

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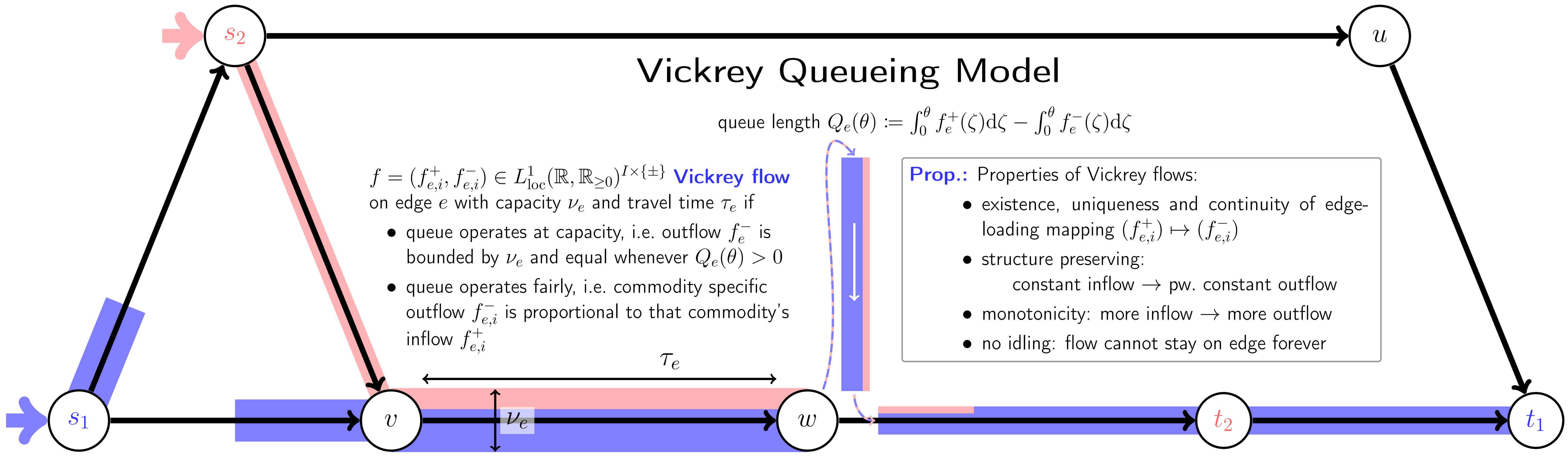
Setting: Car traffic (continuous flow) in road network (directed graph) creates dynamically changing congestion (queues)
Car drivers (flow particles) choose shortest routes towards their destination and adapt while travelling based on current information



Vickrey Queueing Model

queue length $Q_e(\theta) := \int_0^\theta f_e^+(\zeta)\mathrm{d}\zeta - \int_0^\theta f_e^-(\zeta)\mathrm{d}\zeta$

$f = (f_{e,i}^+, f_{e,i}^-) \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}_{\geq 0})^{I \times \{\pm\}}$ **Vickrey flow** on edge e with capacity ν_e and travel time τ_e if

- queue operates at capacity, i.e. outflow f_e^- is bounded by ν_e and equal whenever $Q_e(\theta) > 0$
- queue operates fairly, i.e. commodity specific outflow $f_{e,i}^-$ is proportional to that commodity's inflow $f_{e,i}^+$

Prop.: Properties of Vickrey flows:

- existence, uniqueness and continuity of edge-loading mapping $(f_{e,i}^+) \mapsto (f_{e,i}^-)$
- structure preserving: constant inflow → pw. constant outflow
- monotonicity: more inflow → more outflow
- no idling: flow cannot stay on edge forever

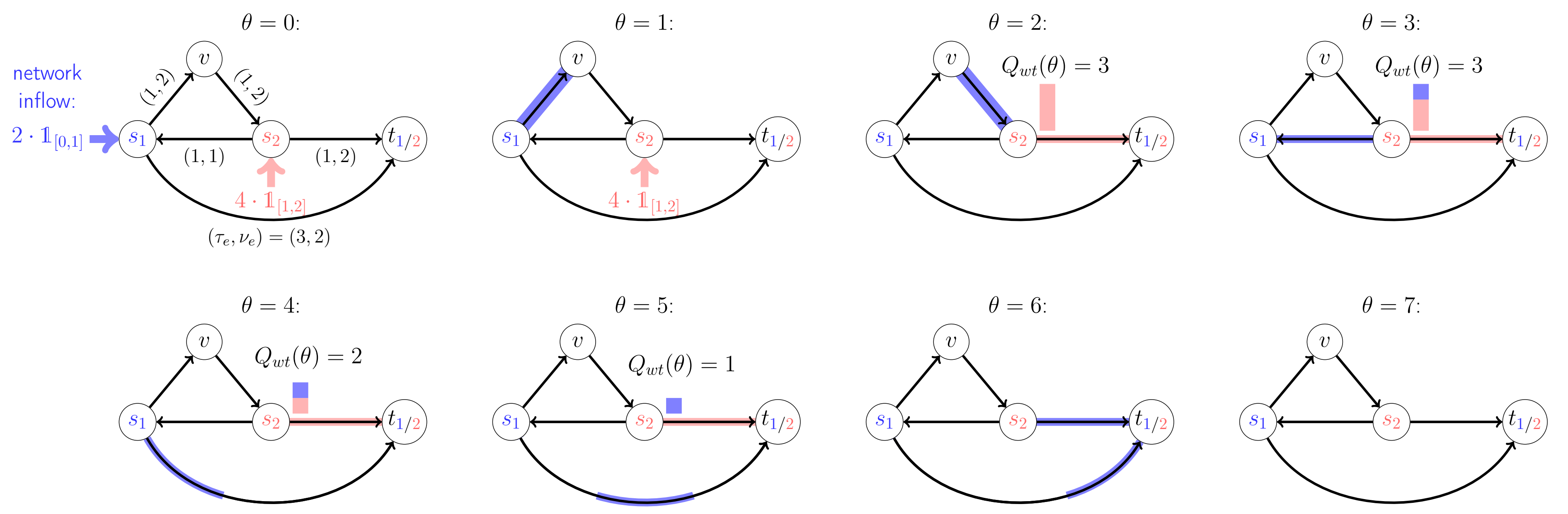
node label $L_{w,i}(\theta) :=$ shortest *current* distance from w to sink t_i of commodity i

Vickrey flow f is **Instantaneous Dynamic Equilibrium (IDE)** if $f_{e,i}^+(\theta) > 0 \implies e = vw$ active at time θ, i.e. $L_{v,i}(\theta) = \tau_e + \frac{Q_e(\theta)}{\nu_e} + L_{w,i}(\theta)$

Existence?

Computation?

Idea: Iteratively extend partial IDE (i.e. an IDE until some fixed time θ) until all flow has reached a sink, e.g.:



This reduces the problem to existence/computation of a single extension and a limit/finiteness argument (+ termination).

Extension-Lemma

Lem.: For any partial IDE (f, θ) and any $\varepsilon > 0$, there exists an IDE-extension $(g, \theta + \varepsilon)$ satisfying
(a) $f = g$ until θ,
(b) flow conservation at nodes during $[\theta, \theta + \varepsilon)$,
(c) Vickrey edge-dynamics during $[\theta, \theta + \varepsilon)$ and
(d) IDE-property during $[\theta, \theta + \varepsilon)$.

Proof: Split into easy and hard constraints:

- $K := \{ (g, \theta + \varepsilon) \text{ satisfies (a) and (b)} \}$
- $\Gamma : K \to 2^K : g \mapsto \left\{ h \in K \;\middle|\; \begin{array}{l} (g^+, h^-) \text{ satisf. (c)} \\ (h^+, g^-) \text{ satisf. (d)} \end{array} \right\}$

Fix point of Γ is IDE-extension and exists. ☐

Limit-Theorem

Thm.: A set $\mathcal{F}$ of partial flows contains some (f, ∞) if
(a) it is non-empty,
(b) any $(f, \theta) \in \mathcal{F}$ has an extension $(g, \theta + \varepsilon) \in \mathcal{F}$
(c) and any ascending sequence has an upper limit.

Proof: Zorn's Lemma (or inductively, if ε is lower bounded) ☐

Existence

Thm.: For multi-commodity networks with p-integrable network inflow and $\tau_e \in \mathbb{R}_{\geq 0}$ IDE are guaranteed to exist.

Proof: Apply Limit-Theorem: (a) satisfied by $(0, 0)$, (b) by Extension-Lemma and (c) because limit of partial IDE is again a partial IDE. ☐

More details in my thesis:
https://lukasmgraf.de/files/phd-thesis.pdf

Extension-Computation

Lem.: In networks with $\tau_e > 0$ and piecewise-constant flow rates, we can compute IDE-extensions. For single-commodity even in polynomial time.

Proof: Formulate as MIP (multi-commodity) or Wardrop equilibrium problem per node (single-commodity). ☐

Bounding the Number of Extensions

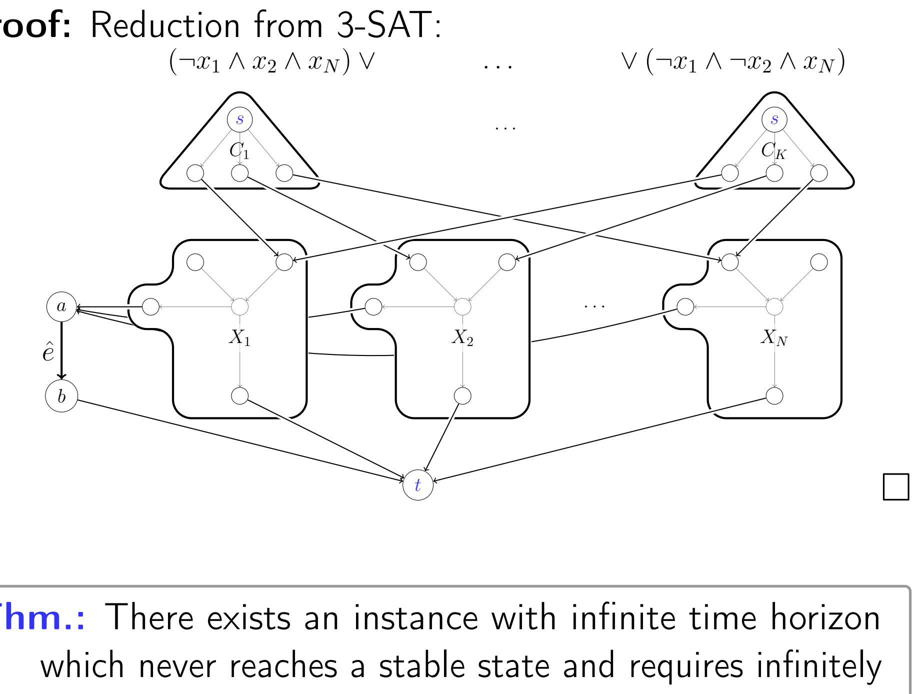
Lem.: For single-commodity: If node-inflow is constant and node-labels closer to the sink are linear, then only finitely many extensions are needed at this node.

Proof: Only finitely many different extensions possible. "Minimal" one started by different edge every time. ☐

Computational Complexity

Thm.: Existence of IDE with special properties (e.g. not using a specific edge $\hat{e}$) is NP-hard.

Proof: Reduction from 3-SAT:



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Thm.: There exists an instance with infinite time horizon which never reaches a stable state and requires infinitely many extension of arbitrarily small length.

Quality?

limited information + selfish behaviour = longer travel times
→ **Question:** Can we bound the makespan of IDE?

Termination for Single-Commodity

"Clear": In acyclic networks, the makespan is bounded by

$$\frac{\text{total flow volume}}{\text{min. edge capacity}} + \text{longest path}$$

In general networks, IDE may use cycles (see left) – but closest flow to the sink cannot. Inductively, this yields:

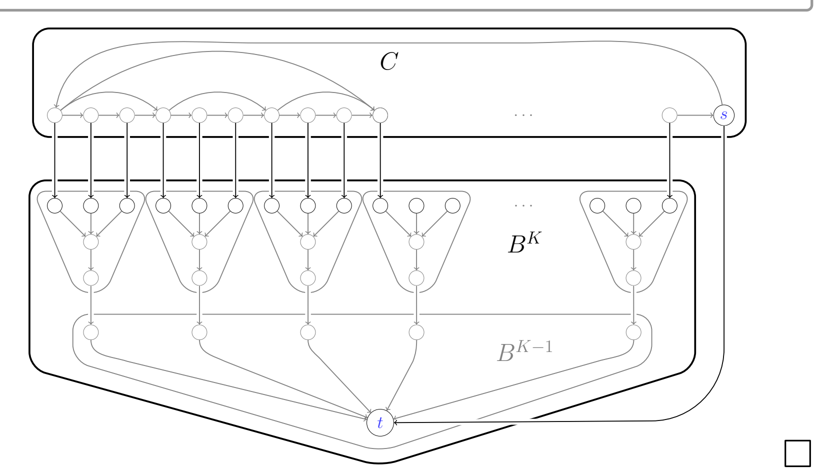
Thm.: For single-commodity, the makespan is bounded by

$$\mathcal{O}\left(\frac{\text{total flow volume}}{\text{min. edge capacity}} \cdot \text{network size}\right).$$

Slow Termination for Single-Commodity

Thm: There exists a single-commodity IDE with makespan $\Omega(\text{total flow volume} \cdot \log(\text{network size}))$

Proof:



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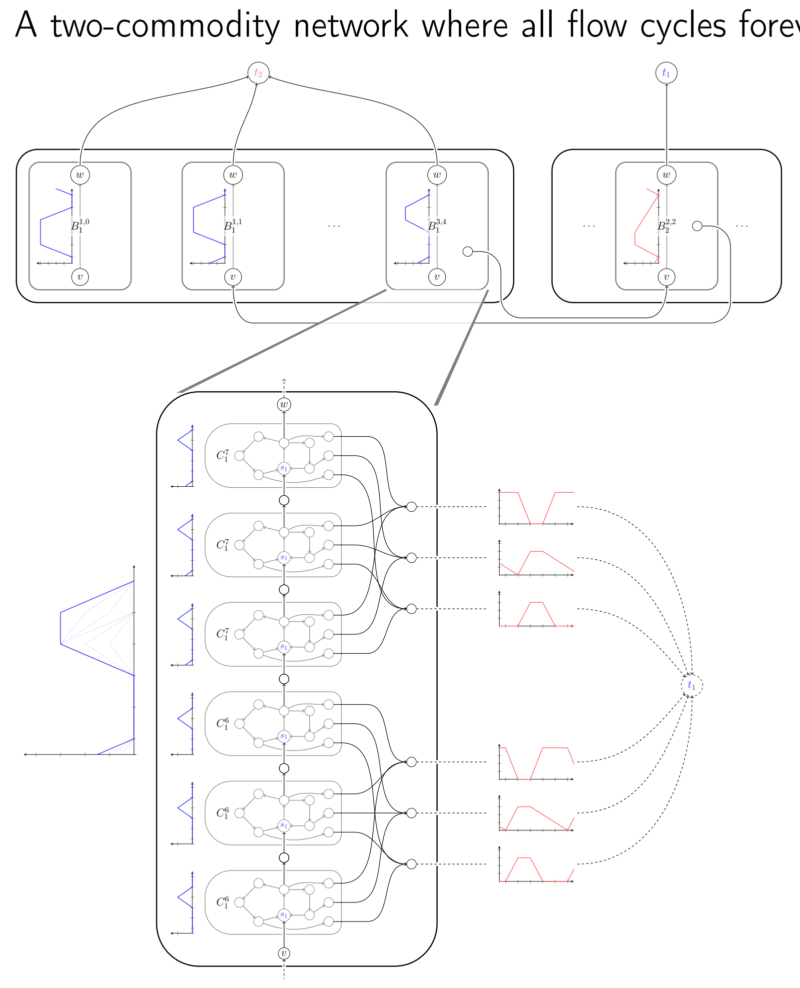
Non-Termination for Multi-Commodity

Thm.: In multi-commodity networks, IDE may cycle forever.

Proof: A two-commodity network where all flow cycles forever:



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